# Digital Communications
# Adaptive Equalization

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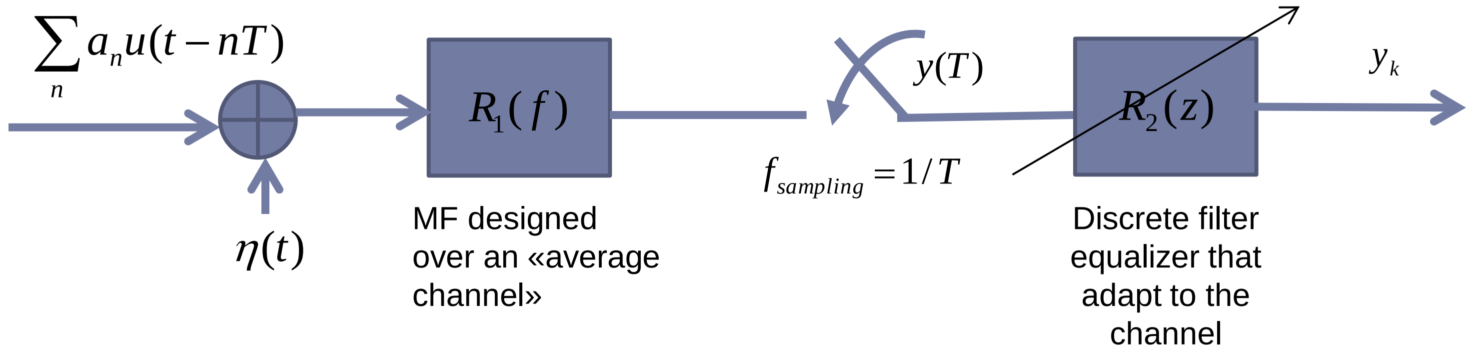
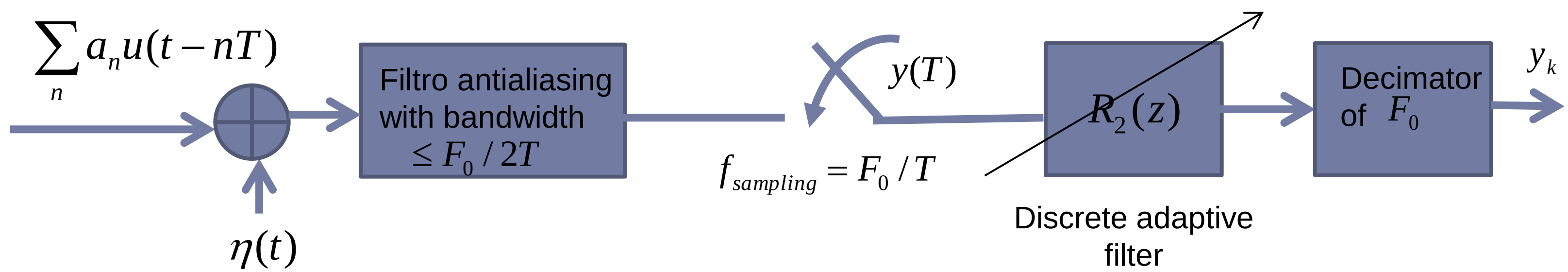
# DIGITAL COMUNICATION SYSTEM
## Adaptive Equalization

When the channel is NOT know or it is time-varying, the receiver must estimate the channel and ADAPT its behavior to it.
Two approaches:

1.



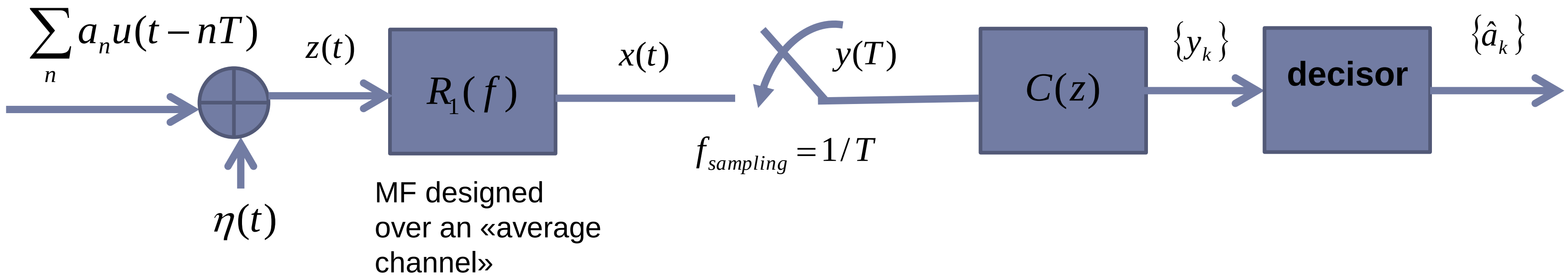

2.



# DIGITAL COMUNICATION SYSTEM
## Adaptive Equalization

Let us consider the approach 1.

1.




$$z(t) = \sum_i a_i u(t-iT) + \eta(t)$$

$$g(t) = u(t) * r_1(t)$$

$$x(t) = \sum_i a_i g(t-iT) + n(t)$$

$$n(t) = \eta(t) * r_1(t)$$

$$C(z) = \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_m z^{-m}$$

Not causal for sake of simplicity $\Rightarrow$ central tap $c_0$ with zero delay

$$C(e^{j2\pi fT}) = T \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_m e^{-j2\pi fmT}$$

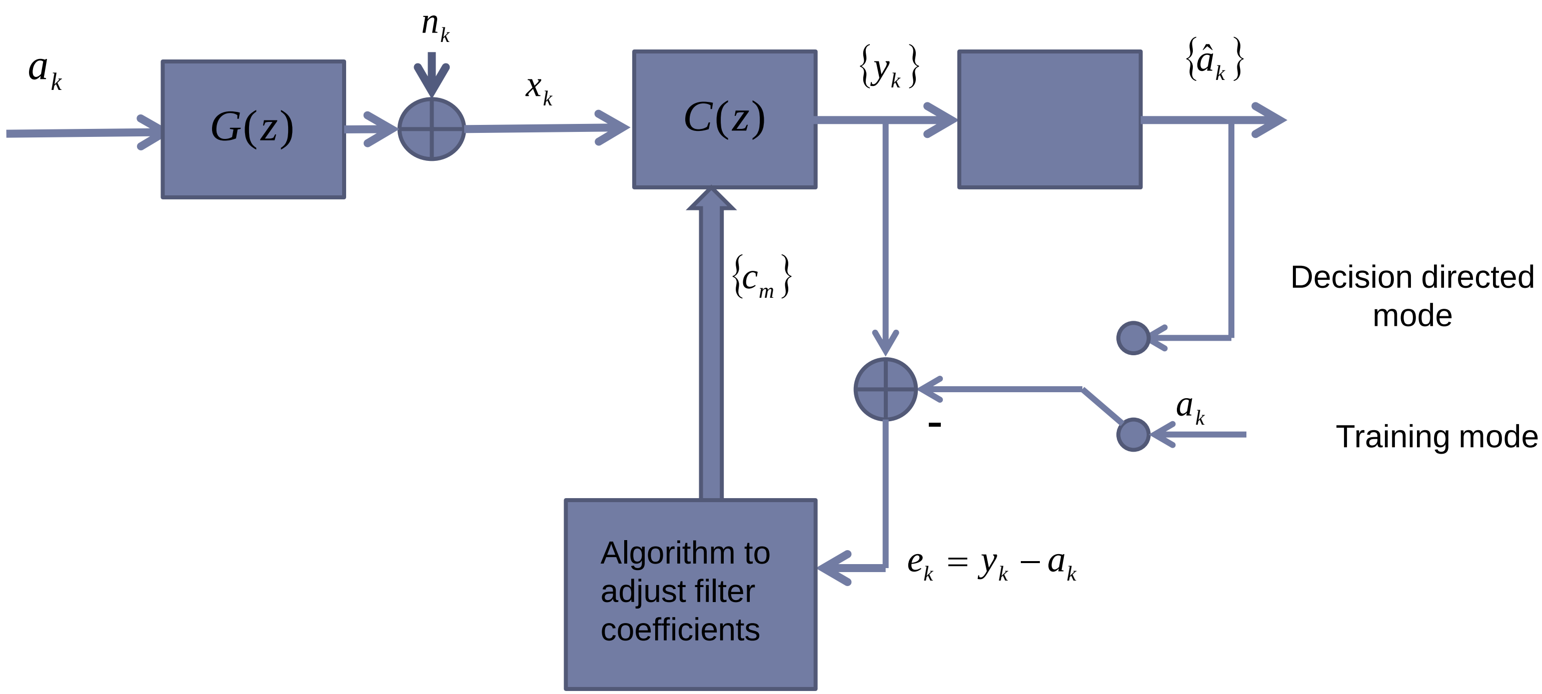
AFTER THE SAMPLER

$$x_k = \sum_i a_i g_{k-i} + n_k \quad \Longleftrightarrow \quad X(z) = A(z)G(z) + N(z)$$

$$y_k = \sum_{m=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_m x_{k-m} \quad \Longleftrightarrow \quad Y(z) = C(z)X(z)$$

# DIGITAL COMUNICATION SYSTEM
## Adaptive Equalization

Discrete equivalent scheme of the receiver




1. Decision directed mode: $e_k = y_k - \hat{a}_k$
2. Training mode: $e_k = y_k - a_k$ in the training sequence, the transmitted symbols are known.

# DIGITAL COMUNICATION SYSTEM
## Adaptive Equalization: strategy

1. Choice of the performance metric $J$ that depends from the coefficient $\{c_m\}$
   e.g.: MSE criteria $J = E\left[|e_k|^2\right]$

2. Choice of the strategy to update the coefficient

e.g. steepest descendent method

$$\underline{c}^{k+1} = \underline{c}^{k} - \mu^{k} \frac{\partial}{\partial \underline{c}^{k}} J$$

It is not easy to calculate that gradient

⇓

It is possible to estimate the gradient from $x_k$ and $e_k$

⇓

$$\boxed{\underline{c}^{k+1} = \underline{c}^{k} - \mu^{k} \widehat{\frac{\partial J}{\partial \underline{c}^{k}}}}$$

# DIGITAL COMUNICATION SYSTEM
## MSE adaptive algorithm

$$J = E\left[\left|e_k\right|^2\right] = E\left[\left|y_k - a_k\right|^2\right] = E\left[\left|\sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_j x_{k-j} - a_k\right|^2\right]$$

$$\boxed{\frac{\partial J}{\partial \underline{c}_l} = 0} \Rightarrow 2E\left[(y_k - a_k)x_{k-l}^*\right] = 0 \Rightarrow \sum_{j=-\frac{N-1}{2}}^{\frac{N-1}{2}} c_j \underbrace{E\left[x_{k-j}x_{k-l}^*\right]}_{R_x(l-j)} = \underbrace{E\left[a_k x_{k-l}^*\right]}_{V(l)}$$

$$\underline{R}_x = E\left[\underline{x}_k^* \underline{x}_k^T\right] = \begin{bmatrix} R_x(0) & R_x(-1) & . & R_x(-(N-1)) \\ R_x(1) & R_x(0) & . & . \\ . & . & . & R_x(-1) \\ R_x(N-1) & . & R_x(1) & R_x(0) \end{bmatrix} \qquad y_k = \underline{c}\,\underline{x}_k \qquad \underline{c} = \begin{bmatrix} c_{\frac{N-1}{2}} \\ . \\ c_0 \\ \\ c_{-\frac{N-1}{2}} \end{bmatrix} \qquad \underline{x}_k = \begin{bmatrix} x_{k+\frac{N-1}{2}} \\ . \\ x_k \\ \\ x_{k-\frac{N-1}{2}} \end{bmatrix}$$

Autocorrelation matrix of
the sampled PAM signal

$$\underline{V} = E\left[a_k \underline{x}_k^*\right]$$

# DIGITAL COMUNICATION SYSTEM
## MSE adaptive algorithm

$$J = \underline{c}^{*T}\underline{R}_x\underline{c} - 2\,\mathrm{Re}\left[\underline{c}^{*T}\underline{V}\right] + \alpha_0$$

Properties of the matrix $R_x$

1) It is Hermitian: $R_x^{*T} = R_x$ $\Rightarrow$ $\mathrm{Re}\left[R_x\right] = \mathrm{Re}\left[R_x^T\right]$, $\mathrm{Im}\left[R_x\right] = \mathrm{Im}\left[R_x^T\right]$

2) It is Toeplix: all diagonal have the same element

3) It is semi-definite positive: $\underline{x}^{*T}R_x\underline{x} \geq 0 \quad \forall \underline{x}$

$\underline{c} = \underline{c}_{\mathrm{Re}} + j\underline{c}_I \qquad \underline{V} = \underline{V}_{\mathrm{Re}} + j\underline{V}_I \qquad \underline{R_x} = \underline{R_x}_{\mathrm{Re}} + j\underline{R_x}_I$

Gradient with respect to a vector:

$$\nabla_{\underline{c}} = \nabla_{\underline{c}_{\mathrm{Re}}} + j\nabla_{\underline{c}_I}$$

## DIGITAL COMUNICATION SYSTEM
## MSE adaptive algorithm

$$\nabla_{\underline{c}_{\mathrm{Re}}} \underline{c}^{*T} \underline{R}_x \underline{c} = 2\underline{R}_{x_R} \underline{c}_{\mathrm{Re}} - 2\underline{R}_{x_I} \underline{c}_I$$

$$\nabla_{\underline{c}_I} \underline{c}^{*T} \underline{R}_x \underline{c} = 2\underline{R}_{x_R} \underline{c}_I + 2\underline{R}_{x_I} \underline{c}_{\mathrm{Re}}$$

$$\nabla_{\underline{c}_{\mathrm{Re}}} \mathrm{Re}\left[\underline{c}^{*T} V\right] = V_{\mathrm{Re}}$$

$$\nabla_{\underline{c}_I} \mathrm{Re}\left[\underline{c}^{*T} V\right] = V_I$$

$$\nabla_{\underline{c}} \underline{c}^{*T} \underline{R}_x \underline{c} = 2\underline{R}_x \underline{c}$$

$$\nabla_{\underline{c}} \mathrm{Re}\left[\underline{c}^{*T} V\right] = V$$

$$\nabla_{\underline{c}} J = 2\left(\underline{R}_x \underline{c} - \underline{V}\right)$$

$$\underline{c}_{opt} = \underline{R}_x^{-1} \underline{V}$$

$$J_{\min} = \alpha_0 - \underline{V}^{*T} \underline{c}_{opt}$$

# DIGITAL COMUNICATION SYSTEM
## MSE adaptive algorithm
## Orthogonality principle

when $\underline{c} = \underline{c}_{opt}$ $\Rightarrow$ $e_k \perp x_k$

$$E\left[e_k \underline{x}_k^*\right] = E\left[\underline{x}_k^*(y_k - a_k)\right] = E\left[\underline{x}_k^*\left(\underline{x}_k^T \underline{c}_{opt} - a_k\right)\right] = \underline{R}_x \underline{c}_{opt} - \underline{V} = 0$$

# DIGITAL COMUNICATION SYSTEM
## MSE adaptive algorithm
## Iterative algorithm: steepest descendent method

J has a minimum in c

$$\underline{c}(n+1) = \underline{c}(n) - \mu\left(\frac{1}{2}\nabla_{\underline{c}(n)} J\right)$$

$$\nabla_{\underline{c}(n)} J = 2\left(\underline{R}_x \underline{c}(n) - \underline{V}\right)$$

$$\underline{c}(n+1) = \underline{c}(n) + \mu\left(\underline{V} - \underline{R}_x \underline{c}(n)\right) = \left(\underline{I} - \mu \underline{R}_x\right)\underline{c}(n) + \mu \underline{V}$$

Let us define $\underline{q}(n) = \underline{c}(n) - \underline{c}_{opt}$

$\Downarrow$

$$\underline{q}(n+1) = \left(\underline{I} - \mu \underline{R}_x\right)\underline{q}(n) = \left(\underline{I} - \mu \underline{R}_x\right)^n \underline{q}(0)$$

# DIGITAL COMUNICATION SYSTEM
## MSE adaptive algorithm
## Iterative algorithm: Least Mean Square (or stocastic gradient)

In many cases of interest, it is not possible to know the autocorrelation matrix $R_x$ and the previous method is not applicable

In the LMS algorithm, the statistic average is replaced by the time average

$$\min_{\underline{c}} |e_k|^2 = |y_k - a_k|^2 = \left|\underline{c}^T \underline{x}_k - a_k\right|^2 =$$
$$= \underline{c}^{*T} \underline{x}_k^* \underline{x}_k^T \underline{c} - 2\,\mathrm{Re}\left[a_k \underline{c}^{*T} \underline{x}_k^*\right] + |a_k|^2$$

$$\nabla_{\underline{c}} |e_k|^2 = 2 e_k \underline{x}_k^*$$

## DIGITAL COMUNICATION SYSTEM
## MSE adaptive algorithm
## Iterative algorithm: Least Mean Square (or stocastic gradient)

$$\underline{c}_{k+1} = \underline{c}_k - \mu\left(\frac{1}{2}\nabla_{\underline{c}_k}\left|e_k\right|^2\right)$$

$$\Downarrow$$

$$\underline{c}_{k+1} = \underline{c}_k - \mu e_k \underline{x}_k^*$$

$$\Downarrow$$

$$\underline{c}_{k+1}(j) = \underline{c}_k(j) - \mu e_k x_{k-j}^* \qquad -\frac{N-1}{2} \le j \le \frac{N-1}{2}$$

In the previous method, the index was the index of the iteration to solve a system of linear equations. Now, the index is the TIME, it is the number of the sample in input to the filter. Each iteration is a new sample in input. The algorithm work on the temporal average to estimate the gradient.